Math 010 Exam 3
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. (6 pts) Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathrm{R} \right\}$ with usual addition, but scalar multiplication defined by $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cy \end{bmatrix}$.

Though several vector space axioms hold, this is not a vector space. Show that one of the following axioms fails.

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$.

#8 LHS: $(k+m)\vec{u} = \begin{bmatrix} x \\ (k+m)y \end{bmatrix} = \begin{bmatrix} x \\ km+ym \end{bmatrix}$

RHS: $k\vec{u} + m\vec{u} = k\begin{bmatrix} x \\ y \end{bmatrix} + m\begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x \\ ky \end{bmatrix} + \begin{bmatrix} x \\ my \end{bmatrix} = \begin{bmatrix} 2x \\ ky+my \end{bmatrix}$

Since $\begin{bmatrix} x \\ ky+my \end{bmatrix} \neq \begin{bmatrix} 2x \\ ky+my \end{bmatrix}$ for all x,

that is, LHS $\neq$ RHS, axiom 8 fails.

2. (6 pts) Use the Subspace Test to determine whether

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathrm{R}^3 : x + y - z = 0 \right\}$$

is a subspace of R^3.

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be in W. $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

$$(u_1 + v_1) + (u_2 + v_2) - (u_3 + v_3)$$
$$= u_1 + u_2 - u_3 + v_1 + v_2 - v_3$$
$$= 0 + 0 \quad \text{since } \vec{u}, \vec{v} \in W.$$
$$= 0. \quad \text{Thus } \vec{u} + \vec{v} \in W.$$

Let k be any scalar.

$k\vec{u} = (ku_1, ku_2, ku_3)$

$$ku_1 + ku_2 - ku_3 = k(u_1 + u_2 - u_3)$$
$$= k(0) = 0$$

$k\vec{u} \in W$.

W is a subspace.

3. (6 pts) Determine whether

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \left\{\begin{bmatrix}2\\0\\2\end{bmatrix},\right. & \begin{bmatrix}2\\1\\3\end{bmatrix}, & \left.\begin{bmatrix}3\\2\\5\end{bmatrix}\right\} \end{array}$$

is linearly independent. If it is dependent, express one vector as a linear combination of the other two.

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since row reducing produces a row of zeros, the set is dependent.

To find a linear combination, find rref.

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} -1/2 \\ 2 \\ 0 \end{matrix}\right] \end{array}$$

Since $\vec{c}_3 = -\frac{1}{2}\vec{c}_1 + 2\vec{c}_2$, we have

$$\vec{v}_3 = -\frac{1}{2}\vec{v}_1 + 2\vec{v}_2 .$$

4. a. (4 pts) Show that the given set is a basis for R^2.

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$$\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} = -1-1 = -2 \neq 0.$$

Thus the set is a basis for R^2.

OR $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \to \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Since this reduces to the identity, the set is a basis.

b. (2 pts) Answer without performing any computations: Is it possible that the given set is a basis for R^3? Briefly explain why or why not.

$\left\{\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}-2\\4\\-1\end{bmatrix}\right\}$

No. A basis for R^3 has 3 vectors. This set cannot span R^3.

5. (6 pts) Determine whether each statement is true or false. Justify briefly.

a. If A and B are row equivalent matrices, they have the same column space.

False. Row operations do not preserve column space. (They preserve row space.)

b. If a set contains the zero vector, then it is linearly dependent.

True. This is because the coefficient of $\vec{0}$ in a linear combination can be nonzero, but the result of the linear combination can still be zero.

c. The intersection of two subspaces of a vector space is always a subspace.

True. We proved a theorem similar to this in class.

6. (8 pts) Find a basis for the subspace of R^3 spanned by

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

State the dimension of the subspace.

$$\begin{bmatrix}1 & 2 & 0\\1 & 3 & 1\\0 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & -1 & -1\\0 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Since columns 1 and 2 are pivot columns, the basis is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\1\end{bmatrix}\right\}$ with dimension 2.

7. (10 pts) Let $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and let S denote the standard basis.

a. Find $P_{B \to S}$.

$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$ shows that $P_{B \to S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

b. Find $[\mathbf{v}]_B$ for $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \qquad [\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

c. Use the matrix you found in part (a) to compute $[\mathbf{v}]_S$. Briefly comment on this result.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [\vec{v}]_S.$$

This is $\vec{v}$, since S is the standard basis for $\mathbb{R}^2$. It verifies that $P_{B \to S}$ is correct.

8. (8 pts) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Find:

a. a basis for $\mathrm{null}(A)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_3$, $x_2 = -2x_3$. Let $x_3 = t$.

$\vec{x} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix}$, so the basis is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

b. $\mathrm{rank}(A)$

2 (the number of leading 1s)

c. $\mathrm{nullity}(A)$

1 (the dimension of the null space)

9. (8 pts) Prove **one** of the following.

a. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r$ in exactly one way.

b. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

a. Let $\vec{v} \in V$. Then since S spans V, there are c_i such that

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. Suppose

there are k_i such that

$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$. Then

$$\vec{v} - \vec{v} = \vec{0} = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) - (k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n)$$

$$\Rightarrow (c_1 - k_1)\vec{v}_1 + (c_2 - k_2)\vec{v}_2 + \cdots + (c_n - k_n)\vec{v}_n = \vec{0}$$

Since S is linearly independent, $c_i = k_i$ for all i. Thus the representation is unique.

b. Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

If $c_3 \neq 0$, then $\vec{v}_3 = -\frac{c_1}{c_3}\vec{v}_1 - \frac{c_2}{c_3}\vec{v}_2$, Contracting that $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$.

If $c_3 = 0$, then $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ so $c_1 = c_2 = 0$ because $\{\vec{v}_1, \vec{v}_2\}$ is an independent set. Thus $c_1 = c_2 = c_3 = 0$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set.